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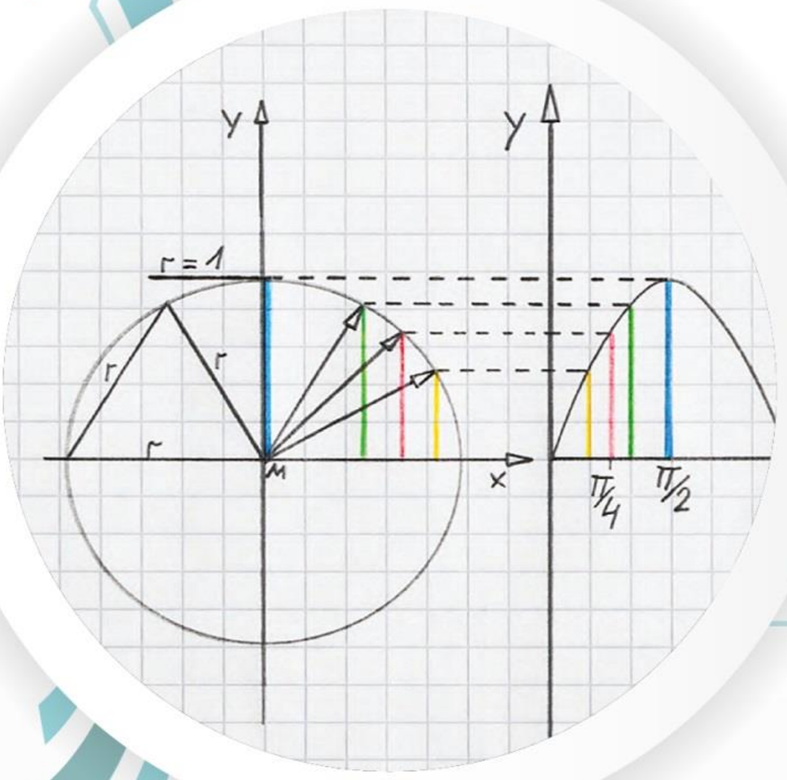


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# Carleman's formula for skew-symmetric matrices

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**Abstract:** In this work, an integral formula in a matrix polydisk is obtained that allows one to recover the value of a function from the Hardy class at any internal point of the matrix polydisk from its values on part of the skeleton. All evidence presented is valid only if it $m$ is even.

**Keywords**

## Introduction

The question of finding Carleman's formulas has long attracted many specialists in complex analysis and mathematical physics. These formulas allow you to solve ill-posed problems of mathematical physics and analysis. For example, reconstruct the values of holomorphic or harmonic functions in a domain from their values on uniqueness sets. The first result in this direction was obtained by T. Carleman in 1926 y. in (see [6]), therefore all formulas of this type are usually called Carleman formulas. These studies were continued by G.M. Goluzin and V.I. Krylov (see [2]), who gave a general method for obtaining Carleman formulas in one-dimensional complex analysis. Their method did not work in the multivariate case. In 1956 y. M.M. Lavrentiev (see [4]) proposed his own method based on approximation of the kernels of integral representations. In the monograph (see [5]) one can find many examples of the construction of Carleman's formulas in various questions of mathematical physics.

Further development of this theory in multidimensional complex analysis can be found in the book by L.A. Aizenberg (see [1]) who constructed the Carleman formula based on the Bochner-Martinelli integral representation in convex (or linearly convex) bounded domains.

In homogeneous domains in, automorphism groups can be used $J^n$ to find such formulas (see [1]). In [3], the case of Siegel domains was considered , i.e. unlimited realizations of homogeneous domains, and Carleman formulas are given that restore the values of holomorphic functions on the skeleton of a Siegel domain.

It is known that formulas in the spirit of Carleman solve the problem of reconstructing a holomorphic function in a domain $D$ that behaves quite well when approaching the boundary $\partial D$, from its values on a certain set of uniqueness $M \subset \partial D$ that does not contain the Shilov boundary.

Let us define a class $H^p(D)\ (p>0)$ as the class of all functions $f$ holomorphic in $D$ for which

$$\sup_{0<r<1} \int_{S(D)} |f(rU)|^p \, d\mu < +\infty,$$

here $rU=(ru_{11},ru_{12},...,ru_{mm})$, and $d\mu$ is the normalized Lebesgue measure on the manifold $S(D)$, invariant under rotations.

Let $\Re_3$ – classical domain of the third type according to the classification of E. Cartan, is defined as a set (see [9])

$$\Re_3=\left\{Z\in \mathrm{J}\,[m\times m]:I+Z\overline{Z}>0\right\},$$

where are $Z$ – skew-symmetric order matrices $m$, ( $I$ – unit matrix).

There are many on the border

$$S_3=\left\{Z\in \mathrm{J}\,[m\times m]:I+Z\overline{Z}=0\right\},$$

which called the skeleton $\Re_3$(notice, that $S_3$ I is the Shilov's boundary for $\Re_3$).

Let the set $M\subset S_3$ have a positive measure: $\mu_1(M)>0$, $\mu_1$ - the Haar measure on $S_3$.

Let's denote

$$M_{0,u}=\left\{\xi:\xi\in M,\xi=\lambda u,|\lambda|=1\right\},u\in SU(m), \tag{1}$$

$$M_0'=\left\{u:u\in SU(m),m_1M_{0,u}>0\right\},$$

where is $SU(m)$ – a group of special unitary matrices $u$, i.e. $\det(u)=1$, $m_1$ – normalized Lebesgue measure on the unit circle $\partial U$. By Fubini's theorem $\mu_0(M_0')>0$, $\mu_0$ – normalized Lebesgue measure on $SU(m)$.

**Lemma 1 (see [7]).** *Haar $d\mu$ measure of a variety $S_3$ can be written in the form*

$$d\mu=h(u)d\varphi d\mu_0(u),$$

*where $d\mu_0$ – the normalized Lebesgue measure is on $SU(m)$, and $h$ – the smooth positive function is on $SU(m)$.*

Next, let

$$\varphi_0=\exp\psi_0,\ \text{Where}\ \psi_0(\xi)=\frac{1}{2\pi i}\int_{M_{0,u}}\frac{\eta+\lambda}{\eta-\lambda}\frac{d\eta}{\eta}.$$

**Lemma 2 (see [1], [7]).** *Let $f\in H^1(\Re_3)$, then the formula is valid*

$$f(0)=\frac{m}{\int_{M_0'}d\mu_0}\lim_{l\to\infty}\int_M f(\xi)\left[\frac{\varphi_0(\xi)}{\varphi_0(0)}\right]^l d\mu_1. \tag{2}$$

Let be $Z=\left(Z^1,Z^2,...,Z^n\right)\in \mathrm{J}^{\,n}\left[m\times m\right]$ – a vector composed of square matrices of order $m$ considered over the field of complex numbers J .

**The Main part**

The matrix unit polydisk $T_n$ in space $\mathrm{J}^{\frac{n(m-1)}{2}}$ is defined as the direct product $\Re_3$:

$$T_n = \Re_3{}^n = \underbrace{\Re_3 \times \ldots \times \Re_3}_{n\ раз} = \left\{ Z = \left(Z^1,\ldots,Z^n\right) : Z^j \in \Re_3, \left(Z^j\right)' = -Z^j, j = 1,\ldots,n \right\}.$$

The skeleton of a matrix unit polydisk is the set

$$\Gamma_n = \underbrace{S_3 \times \ldots \times S_3}_{n\ раз}.$$

Let us consider an automorphism of the matrix unit polydisk $T_n$ that interchanges the points $A \in T_n$ and $0$. Such an automorphism has the form (see [9], art. 86)

$$\Phi_A(Z) = \left(\Phi_A^1(Z^1),\ldots,\Phi_A^n(Z^n)\right),$$

where

$$\Phi_A^j(Z^j) = Q^j(Z^j - A^j)(I + \bar{A}^j Z^j)^{-1}\left(\bar{Q}^j\right)^{-1},\ j = 1,2,\ldots,n,$$

$Q^j - [m \times m]$ matrices satisfying the conditions

$$\bar{Q}^j\left(I + A^j \bar{A}^j\right)\left(Q^j\right)' = I,$$

In particular, when $A = 0$ we get

$$\Phi_0(Z) = (\Phi_0^1(Z^1),\ldots,\Phi_0^n(Z^n)) \text{ and } \Phi_0^j(Z^j) = R^j Z^j \left(\bar{R}^j\right)^{-1}.$$

Let $E = \left(E^1,\ldots,E^n\right) \subset \Gamma_n$ it be $\mu E > 0$. We denote the projection of the skeleton $\Gamma_n$ on space $J^{n-1}[m \times m]$ by $\Gamma_{n-1}$. We denote the points $\Gamma_{n-1}$ by $\breve{\xi} = \left(\xi^2,\ldots,\xi^n\right)$.

Let's define sets

$$E_{0,\xi} = \left\{ Z : Z \in E, \Phi_0^1(Z^1) = \theta, \Phi_0^j(Z^j) = \theta \Phi_0^j(\xi^j), j = \overline{2,n}, \theta \in S_2 \right\},$$

$$\tilde{E}_0 = \left\{ Z \in E : \mu_1 E_{0,\xi} > 0 \right\}.$$

Sets $E_{0.\xi}$ and $\tilde{E}_0$ are subsets with positive set measure $E$ and their Cartesian product coincides with $E$, i.e. $E = E_{0,\xi} \times \tilde{E}_0$.

Let's introduce an auxiliary function $\varphi_0 = \exp\psi_0$, where

$$\psi_0(\xi) = \frac{1}{2\pi i} \int_{E^1_{0,\xi^1}} \frac{\eta + \lambda}{\eta - \lambda} \frac{d\eta}{\eta}.$$

Set $E^1_{0,\xi^1}$ - is defined in the same way as in (1), and its Cartesian product with the set $\tilde{E}_{0,\xi}$ forms the set $E_{0,\xi}$, i.e. $E^1_{0,\xi^1} \times \tilde{E}_{0,\xi} = E_{0,\xi}$.

**Lemma 3.** *Let* $f \in H^1(T_n)$, $E \subset \Gamma_n$, $\mu(E) > 0$. *Then the following formula is valid*

$$f(0) = \frac{m}{\int\limits_{\tilde{E}_0} d\mu_{n-1} \int\limits_{\tilde{E}_{0,\xi}} d\mu_0} \lim_{l \to \infty} \int\limits_E f(Z) \left[\frac{\varphi_0(\xi)}{\varphi_0(0)}\right]^l d\mu. \qquad (3)$$

**Proof.** Let the multitude $E \subset \Gamma_n$ $\mu(E) > 0$. Consider the projection $E$ onto $S_3$, we denote this projection by $E_{0,\xi}$. For the set $E_{0,\xi}$, by Lemma 2 the formula is true

$$f(0) = \frac{m}{\int\limits_{\tilde{E}_{0,\xi}} d\mu_0} \lim_{l\to\infty} \int\limits_{E_{0,\xi}} f(\xi) \left[\frac{\varphi_0(\xi)}{\varphi_0(0)}\right]^l d\mu_1 .$$

Let's integrate both sides of this equality over $\tilde{E}_0$:

$$\int\limits_{\tilde{E}_0} f(0) d\mu_{n-1} = \int\limits_{\tilde{E}_0} \left( \frac{m}{\int\limits_{\tilde{E}_{0,\xi}} d\mu_0} \lim_{l\to\infty} \int\limits_{E_{0,\xi}} f(\xi) \left[\frac{\varphi_0(\xi)}{\varphi_0(0)}\right]^l d\mu_1 \right) d\mu_{n-1} .$$

Because

$$\left| \int\limits_{E_{0,\xi}} f(\xi) \left[\frac{\varphi_0(\xi)}{\varphi_0(0)}\right]^l d\mu_1 \right| \le \left| \int\limits_{S_3} f(\xi) \left[\frac{\varphi_0(\xi)}{\varphi_0(0)}\right]^l d\mu_1 \right| + \left| \int_{S_3 \setminus E_{0,\xi}} f(\xi) \left[\frac{\varphi_0(\xi)}{\varphi_0(0)}\right]^l d\mu_1 \right| \le$$

$$\le |f(0)| + \int\limits_{S_3} |f(\xi)| d\mu_1$$

the integrand is limited. From here and by Lebesgue's theorem (see [8]) we have the following

$$f(0) = \frac{m}{\int\limits_{\tilde{E}_0} d\mu_{n-1} \int\limits_{\tilde{E}_{0,\xi}} d\mu_0} \lim_{l\to\infty} \int\limits_{\tilde{E}_0} \left( \int\limits_{E_{0,\xi}} f(\xi) \left[\frac{\varphi_0(\xi)}{\varphi_0(0)}\right]^l d\mu_1 \right) d\mu_{n-1} .$$

By definition, the set , hence $E = E_{0,\xi} \times \tilde{E}_0$, by Fubini's theorem (see [8]), from the last equality we obtain

$$f(0) = \frac{m}{\int\limits_{\tilde{E}_o} d\mu_{n-1} \int\limits_{\tilde{E}_{0,\xi}} d\mu_0} \lim_{l\to\infty} \int\limits_{E} f(Z) \left[\frac{\varphi_0(\xi)}{\varphi_0(0)}\right]^l d\mu .$$

Here, as in [7], we took advantage of the fact that the function is a $\varphi_0(Z)-$ "quenching" function, and the Cauchy-Szego formula is true for function b $\Re_3$. Therefore, passage to the limit under the integral sign is possible here by virtue of Lebesgue's theorem. Lemma 2 is proven.

Formula (3) restores the value of the function $f$ at a point 0 from its values on $E$. Now, using this formula, we will prove a formula that restores the value $f$ at an arbitrary point in the domain $T_n$.

Let us denote $\grave{\xi} = (\xi^2, ..., \xi^n) \in \Gamma_{n-1}$ and consider the set

$$E_{A,\xi} = \left\{ Z : Z \in E, \Phi_A^1(Z^1) = \theta, \Phi_A^j(\xi^j) = \theta \Phi_A^j(\xi^j), j = \overline{2,n}, \theta \in S_2 \right\}.$$

This set is measurable with respect to the measure $\mu_1$ for almost all $A$ and $\grave{\xi}$. We denote the set $\left\{ \grave{\xi} : \grave{\xi} \in \Gamma_{n-1}, \mu_1 E_{A,\xi} > 0 \right\}$ by $\tilde{E}_A$. It's obvious that $E_{A,\xi} \times \tilde{E}_A = E$. From Fubini's theorem (see [8]) it follows that the Lebesgue $\left( \frac{(m-1)(n-1)}{2} \right)$–measure of this set is positive.

Similar to the previous one, we introduce an auxiliary function

$$\varphi_A = \exp \psi_A,\ \psi_A(\xi) = \frac{1}{2\pi i} \int\limits_{E^1_{A^1,w^1}} \frac{\eta + \lambda}{\eta - \lambda} \frac{d\eta}{\eta},$$

here

$$E_{A^1,W^1} = \left\{ \xi^1 \in E^1, \xi = \left( \Phi^1_A \right)^{-1} \left( \lambda \left( \Phi^1_A \right)^{-1}(w) \right), |\lambda| = 1 \right\},\ W \in \Phi^1_A(SU(m)).$$

**Theorem.** *Let* $f \in H^1(T_n)$, $E \subset \Gamma_n$, $\mu(E) > 0$. *Then for any point* $A \in T_n$ *the formula is true*

$$f(A) = \frac{m}{\mu_{\frac{(m-1)(n-1)}{2}}(\Phi_A^{-1}(\tilde{E}_A))\mu\left(\Phi_A^{-1}(\tilde{E}_{A,\xi})\right)} \times$$

$$\times \lim_{l \to \infty} \int\limits_E f(Z) \left[ \frac{\varphi_A(Z)}{\varphi_A(A)} \right]^l \prod_{j=1}^{n} H\left( A^j, \bar{Z}^j \right) d\mu(Z), \qquad (4)$$

*where*

$$H(A^j, \bar{Z}^j) = \frac{1}{\det(I^{(m)} + A^j \bar{Z}^j)^{\frac{m-1}{2}}}$$

*there is a Cauchy- Szegé kernel for a classical domain of the third type.*

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